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VOLUME 7. INTERPOLATION AND APPROXIMATION

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RECEIVED: 730618.

BRIEF DESCRIPTION:

NEWTON CALCULATES THE COEFFICIENTS OF THE NEWTON POLYNOMIAL THROUGH GIVEN INTERPOLATION POINTS AND CORRESPONDING FUNCTION VALUES.

KEYWORDS:

NEWTON INTERPOLATION,
POLYNOMIAL COEFFICIENTS,
DIVIDED DIFFERENCES.

CALLING SEQUENCE:

THE HEADING OF THE PROCEDURE READS:
"PROCEDURE" NEWTON(N,X,F);
"VALUE"N;"INTEGER"N;"ARRAY"X,F;

THE MEANING OF THE FORMAL PARAMETERS IS:
N: ‹ARITHMETIC EXPRESSION›;
THE DEGREE OF THE POLYNOMIAL;
X: ‹ARRAY IDENTIFIER›;
"ARRAY"X[0:N];
THE INTERPOLATION POINTS;
F: ‹ARRAY IDENTIFIER›;
"ARRAY"F[0:N];
ENTRY: THE POLYNOMIAL VALUES AT THE INTERPOLATION POINTS;
EXIT: THE COEFFICIENTS OF THE NEWTON POLYNOMIAL.

PROCEDURES USED: NONE.

RUNNING TIME: THE NUMBER OF DIVISIONS IS N(N+1)/2.

LANGUAGE: ALGOL 60.

METHOD AND PERFORMANCE:

THE COEFFICIENTS OF THE NEWTON POLYNOMIAL ARE CALCULATED BY INTERPOLATION AT THE GIVEN ARGUMENTS AND FUNCTION VALUES; THE RESULTING SET OF EQUATIONS IS SOLVED BY TRANSFORMING THE CORRESPONDING LOWER TRIANGULAR MATRIX TO DIAGONAL FORM.

EXAMPLE OF USE:

```
"BEGIN" "ARRAY" X,F[0:2];
    "PROCEDURE"NEWTON(N,X,F);
 "CODE"36010;
    X[0]:=0;X[1]:=.5;X[2]:=1;
    F[0]:=1;F[1]:=F[2]:=0;
    NEWTON(2,X,F);
    OUTPUT(61,"("/,"("THE NEWTON COEFF. ARE")",
        /,3(N)")",F[0],F[1],F[2]);
"END"TSTNEWTON;
```

```
THE NEWTON COEFF. ARE
+1.000000000000"+000  -2.000000000000"+000  +2.000000000000"+000
```

SOURCE TEXT(S):

```
"CODE"36010;
"PROCEDURE" NEWTON(N,X,F);
"VALUE" N; "INTEGER" N; "ARRAY" X,F;
"COMMENT" NEWTON DETERMINES THE COEFFICIENTS C[J],J=0,...N,
OF THE INTERPOLATIONPOLYNOMIAL C[0] + C[1] *(X-X[0])+...+
C[N] * (X-X[0])*...*(X-X[N-1]) OUT OF N+1 LIN. EQUAT.
THE ARGUMENTS AND FUNCTIONVALUES MUST BE GIVEN IN
ARRAY X, F[0:N]. THE ARRAY F IS OVERWRITTEN BY
THE COEFFICIENTS C[J],J=0,...N;
"BEGIN" "INTEGER" K,I,IM1;
"REAL" XIM1,FIM1;
IM1:=0;
"FOR" I:= 1 "STEP" 1 "UNTIL" N "DO"
"BEGIN" FIM1:=F[IM1];XIM1:=X[IM1];
    "FOR" K:= I "STEP" 1 "UNTIL" N "DO" F[K]:= (F[K]-FIM1)/(X[K]-XIM1);
    IM1:= I
"END"
"END" NEWTON;
        "EOP"
```

AUTHOR: C.G. VAN DER LAAN.

INSTITUTE: MATHEMATICAL CENTRE.

RECEIVED: 730831.

BRIEF DESCRIPTION:

THIS SECTION CONTAINS THREE PROCEDURES:

INI SELECTS A (SUB)SET OF INTEGERS OUT OF A GIVEN SET OF INTEGERS.

SNDREMEZ EXCHANGES AT MOST N+1 NUMBERS WITH NUMBERS OJT OF A REFERENCE SET.

MINMAXPOL CALCULATES THE COEFFICIENTS OF THE POLYNOMIAL (IN THE GRUNERT FORM) THAT APPROXIMATES A FUNCTION, GIVEN FOR DISCRETE ARGUMENTS, IN SUCH A WAY THAT THE INFINITY NORM OF THE ERROR VECTOR IS MINIMISED.

KEYWORDS:

ARGUMENTS EXTREME VALUES,
CHEBYSHEV POLYNOMIAL,
SECOND REMEZ ALGORITHM,
MINIMAX POLYNOMIAL APPROXIMATION.

SUBSECTION : INI.

CALLING SEQUENCE:

THE HEADING OF THE PROCEDURE READS:
"PROCEDURE" INI(N,M,S);
"VALUE"N,M;"INTEGER"N,M""INTEGER""ARRAY"S;
"CODE" 36020;

THE MEANING OF THE FORMAL PARAMETERS IS:
N,M: <ARITHMETIC EXPRESSION>;

THE NUMBER OF POINTS TO BE SELECTED EQUALS N+1;
THE REFERENCE SET CONTAINS THE NUMBERS 0,1,...,M,M>=N;
S: <ARRAY IDENTIFIER>;
"INTEGER" "ARRAY" S[0:N];
EXIT:THE SELECTED INTEGERS ARE DELIVERED IN S.

PROCEDURES USED: NONE.

LANGUAGE: ALGOL 60.

METHODS AND PERFORMANCE:
THE ARGUMENTS FOR WHICH THE CHEBYSHEV POLYNOMIAL OF DEGREE N ATTAINS ITS EXTREME VALUES ON THE INTERVAL [-1,1] ARE TRANSFORMED TO THE INTERVAL [0,M] BY A LINEAR TRANSFORMATION;FINALLY THE NUMBERS ARE PROPERLY ROUNDED.

REFERENCES : SEE MINIMAXPOL (THIS SECTION).

EXAMPLE OF USE:

```
"BEGIN""INTEGER""ARRAY"S[0:2];
"PROCEDURE"INI(N,M,S);"CODE"36020;
INI(2,20,S);
OUTPUT(61,"("""("INI SELECTS OUT OF 0,1,...,20 THE NUMBERS:")",/,
   3(B-ZDB)")",S[0],S[1],S[2])
"END"
```

```
INI SELECTS OUT OF 0,1,...,20 THE NUMBERS:
   0    10    20
```

SUBSECTION : SNDREMEZ.

CALLING SEQUENCE:

THE HEADING OF THE PROCEDURE READS:

```
"PROCEDURE"SNDREMEZ(N,M,S,G,EM);
"VALUE"N,M;"INTEGER"N,M;"INTEGER""ARRAY"S;"ARRAY" G,EM;
"CODE" 36021;
```

THE MEANING OF THE FORMAL PARAMETERS IS:

N,M:<ARITHMETIC EXPRESSION>;
THE NUMBER OF POINTS TO BE EXCHANGED IS SMALLER THAN OR EQUAL TO N+1;THE REFERENCE SET CONTAINS THE NUMBERS 0,1,...,M ,M>=N;

S: <ARRAY IDENTIFIER>;
"INTEGER" "ARRAY" S[0:N];
ENTRY:IN S ONE MUST GIVE N+1 (STRICTLY) MONOTONE INCREASING NUMBERS OUT OF 0,...,M;
EXIT :N+1 (STRICTLY) MONOTONE INCREASING NUMBERS OUT OF THE NUMBERS 0,1,...,M;

G: <ARRAY IDENTIFIER>;
"ARRAY" G[0:M];
ENTRY:IN ARRAY G[0:M] ONE MUST GIVE FUNCTIONVALUES;

EM: <ARRAY IDENTIFIER>;
"ARRAY" EM[0:1];
ENTRY:0<EM[0]<=G[I],I=0,...,M;
EXIT: EM[1]:=INFINITY NORM OF ARRAY G[0:M].

PROCEDURES USED:ABSMAXVEC=CP31060.

LANGUAGE: ALGOL 60.

METHOD AND PERFORMANCE:

THE SECOND REMEZ ALGORITHM IS USED (MEINARDUS,G.(1964)).

REFERENCES : SEE MINMAXPOL (THIS SECTION).

EXAMPLE OF USE:

```
"BEGIN""ARRAY"EM[0:1],G[0:7];"INTEGER""ARRAY"S[0:2];
"PROCEDURE"SNDREMEZ(N,M,S,G,EM);"CODE"36021;
G[0]:=10;G[1]:=12;G[2]:=-15;G[3]:=-10;
G[4]:=-14;G[5]:=15;G[6]:=10;G[7]:=11;
EM[0]:=10;S[0]:=0;S[1]:=3;S[2]:=6;
OUTPUT(61,"("""("THE NUMBERS:")",/,"("S[J]:")",3(B-D),/,
   "("G[S[J]]:")",3(B-DD)")",
   S[0],S[1],S[2],G[S[0]],G[S[1]],G[S[2]]);
SNDREMEZ(2,7,S,G,EM);
OUTPUT(61,"("//,"("ARE EXCHANGED WITH:")",/,"("S[J]:")",3(B-D),/,
   "("G[S[J]]:")",3(B-DD),//,
   "("THE REFERENCE SET OF FUNCTIONVALUES IS:")",/,8(B-DD)")",
   S[0],S[1],S[2],G[S[0]],G[S[1]],G[S[2]] ,
   G[0],G[1],G[2],G[3],G[4],G[5],G[6],G[7])
"END"
```

```
THE NUMBERS:
S[J]:   0   3   6
G[S[J]]:   10 -10   10

ARE EXCHANGED WITH:
S[J]:   0   2   5
G[S[J]]:   10 -15   15

THE REFERENCE SET OF FUNCTIONVALUES IS:
  10   12 -15 -10 -14   15   10   11
```

SUBSECTION : MINMAXPOL.

CALLING SEQUENCE:

```
THE HEADING OF THE PROCEDURE READS:
"PROCEDURE"MINMAXPOL(N,M,Y,FY,CO,EM);
"VALUE"N,M;"INTEGER"N,M;"ARRAY"Y,FY,CO,EM;
"CODE" 36022;

THE MEANING OF THE FORMAL PARAMETERS IS:
N:      <ARITHMETIC EXPRESSION>;
        THE DEGREE OF THE APPROXIMATING POLYNOMIAL IS N-1;
M:      <ARITHMETIC EXPRESSION>;
        THE NUMBER OF REFERENCE FUNCTION VALUES VIZ. ARGUMENTS
        IS M+1;
Y,FY:   <ARRAY IDENTIFIERS>;
        "ARRAY"Y,FY[0:M];
        ENTRY: FY[I] IS THE FUNCTION VALUE AT Y[I], FOR I=0,...M;
CO:     <ARRAY IDENTIFIER>;
        "ARRAY"CO[0:N];
        EXIT: THE COEFFICIENTS OF THE APPROXIMATING POLYNOMIAL,
              OF DEGREE N-1,ARE DELIVERED IN CO[0:N-1]
              (CO[N-1] IS COEFFICIENT OF Y**(N-1));
   EM:      <ARRAY IDENTIFIER>;
            "ARRAY"EM[0:3];
            ENTRY:  EM[2]:THE MAXIMUM ALLOWED NUMBER OF ITERATIONS;
            EXIT:   EM[0]:THE DIFFERENCE OF THE GIVEN FUNCTION AND
                          THE POLYNOMIAL IN THE FIRST APPROXIMATION
                          POINT;
                    EM[1]:THE INFINITY NORM OF THE ERROR OF
                          APPROXIMATION OVER THE DISCRETE INTERVAL;
                    EM[3]:THE NUMBER OF ITERATIONS PERFORMED.


PROCEDURES USED:
    ELMVEC    =CP34020,
    ABSMAXVEC=CP31060,
    NEWTON    =CP36010,
    POL       =CP31040,
    NEWGRN    =CP31050,
    INI       =CP36020,
    SNDREMEZ  =CP36021.


REQUIRED CENTRAL MEMORY : ONE INTEGER ARRAY AND TWO  REAL ARRAYS OF
    ORDER N + 1, AND ONE REAL ARRAY OF ORDER M + 1  ARE USED.

RUNNING TIME:
    THE SECOND REMEZ ALGORITHM(ON A DISCRETE SET) IS QUADRATIC
    CONVERGENT;IN EACH ITERATION THE NUMBER OF OPERATIONS
    (MULTIPLICATIONS AND ADDITIONS) IS PROPORTIONAL TO M*N.

LANGUAGE: ALGOL 60.
```

METHOD AND PERFORMANCE: SEE REF.[1],C1.7.

REFERENCES:

[1].G.MEINARDUS.
APPROXIMATION OF FUNCTIONS AND THEIR NUMERICAL TREATMENT
(GERMAN).SPRINGER TRACTS IN NATURAL PHILOSOPHY.
VOLUME 4.1964.

EXAMPLE OF USE:

```
"BEGIN""INTEGER"N;
"PROCEDURE"MINMAXPOL(N,M,Y,FY,CO,EM); "CODE" 36022;

 "PROCEDURE" COMPUTE(N,A,B,F);
 "VALUE" N,A,B;"INTEGER" N;"REAL" A,B;
 "REAL" "PROCEDURE" F;
 "BEGIN" "INTEGER" K,L,M;
"REAL"R,T,IDM;
 "ARRAY" COEF[0:N],EM[0:3];
 EM[2]:=10*N;
 M:=100*N;
 "BEGIN" "ARRAY" Y,FY[0:M];
 IDM:=(B-A)/M;
 R:=Y[0]:=A;FY[0]:=F(R);
 R:=Y[M]:=B;FY[M]:=F(R);
 L:=M-1;
 "FOR"K:=1"STEP"1"UNTIL"L"DO"
 "BEGIN"R:=Y[K]:=A+K*IDM;FY[K]:=F(R) "END";
 MINMAXPOL(N,M,Y,FY,COEF,EM);
OUTPUT(61,"(""("COEF:")"/")");
"FOR"K:=0"STEP"1"UNTIL"N-1"DO"OUTPUT(61,"(" ")",COEF[K]);
OUTPUT(61,"("/8S/,2(N),2(B+3ZDB),/")","("EM[0:3]")",EM[0],EM[1],
EM[2],EM[3]);
 "END";
 "END" COMPUTE;

"REAL""PROCEDURE"F(X);"VALUE"X;"REAL"X;
F:=1/(X-10);

 "FOR" N:=2"DO"
"BEGIN" OUTPUT(61,"("//,"("DEGREE=")",D//")",N-1);
    COMPUTE(N,-1,1,F)
"END"
 "END";


DEGREE=1

COEF:
-1.0050378153393"-001   -1.0101010101010"-002
EM[0:3]
-5.0631947616870"-004   +5.0631947616870"-004      +20      +3
```

SOURCE TEXT(S) :

```
"CODE"36020;
"PROCEDURE" INI(N,M,S);
"VALUE" N,M;"INTEGER" N,M;
"INTEGER" "ARRAY" S;
 "COMMENT" INI DELIVERS (MONOTONE) THE ROUNDED VALUES
 OF THE ARGUMENTS,WHERE THE CHEBYSHEV POLYNOMIAL
   OF DEGREE N(TRANSFORMED TO THE INTERVAL [0,M],M>=N)
 ATTAINS ITS MAXIMUM VALUES,
 IN INTEGER ARRAY S[0:N];
"BEGIN""INTEGER"I,J,K,L;"REAL"PIN2;
    PIN2:=ARCTAN(1)*2/N;
    K:=0;L:=N-1;J:=S[0]:=0;S[N]:=M;
 "FOR" K:=K+1 "WHILE" K <  L "DO"
    "BEGIN"I:=SIN(K*PIN2)**2*M;
          J:=S[K]:="IF"I<=J"THEN"J+1"ELSE"I;
          S[L]:=M-J;L:=L-1
"END"K;
"IF"L*2=N"THEN"S[L]:=M/2;
 "END" INI;
          "EOP"
```

```
"CODE"36021;
 "PROCEDURE" SNDREMEZ(N,M,S,G,EM);
 "VALUE" N,M;"INTEGER" N,M;
 "INTEGER" "ARRAY" S; "ARRAY" G,EM;
 "COMMENT" SNDREMEZ EXCHANGES ATMOST N+1 NUMBERS ,GIVEN IN
           INTEGER ARRAY S[0:N], WITH NUMBERS OUT OF THE
           REFERENCE SET 0,...M, UNDER THE CONDITIONS:
      I.   THE ALTERNANCE PROPERTY OF THE FUNCTIONVALUES G[S[J]],
           J=0,...N IS PRESERVED.
      II.  !G[S[J]]!>=!EM[0]!,J=0,...N.
      III. THE FIRST INDEX K , WITH G[K]=INFINITY NORM OF G,
           IS ONE OF THE RESULTING NUMBERS S[0],...S[N].
           IN ARRAY G[0:M] ONE MUST GIVE ERROR FUNCTION VALUES.
           MOREOVER,
           EM[1]:=INFINITY NORM OF G,
      THE PROCEDURE ABSMAXVEC IS USED;
 "BEGIN" "INTEGER" S0,SN,SJP1,I,J,K,UP,INDEXMAX,LOW,NM1;
 "REAL" MAX,MSJP1,HI,HJ,HE,ABSE,H;
 "REAL" "PROCEDURE" ABSMAXVEC(K,LOW,UP,A); "CODE" 31060;
                                                "COMMENT"
```

```
      INDEX MAX:=S0:=SJP1:=S[0];
      HE:=EM[0];LOW:=S0+1;
      MAX:=MSJP1:=ABSE:=ABS(HE);
      NM1:=N-1;
      "FOR" J:= 0 "STEP" 1 "UNTIL" NM1 "DO"
      "BEGIN"
         UP:= S[J+1]-1;
         H:= ABSMAXVEC(I,LOW,UP,G);
         "IF" H > MAX  "THEN" "BEGIN" MAX:= H; INDEX MAX:= I "END";
         "IF" H > ABSE "THEN"
         "BEGIN" "IF" HE * G[I] > 0 "THEN"
            "BEGIN" S[J]:= "IF" MSJP1 < H "THEN" I "ELSE" SJP1;
               SJP1:= S[J+1]; MSJP1:= ABSE
            "END" "ELSE"
            "BEGIN" S[J]:= SJP1; SJP1:= I; MSJP1:= H "END"
         "END" "ELSE"
         "BEGIN" S[J]:=SJP1; SJP1:=S[J+1]; MSJP1:= ABSE "END";
         HE:=-HE;LOW:=UP+2;
   "END" FOR J; SN:= S[N]; S[N]:= SJP1;

   HI:=ABSMAXVEC(I,0,S0-1,G);
   HJ:=ABSMAXVEC(J,SN+1,M,G);
   "IF" J  > M  "THEN" J:=M;
   "IF" HI > HJ "THEN"
   "BEGIN" "IF" HI > MAX "THEN" "BEGIN" MAX:= HI; INDEXMAX:= I "END";
      "IF" SIGN(G[I]) =  SIGN(G[S[0]]) "THEN"
      "BEGIN" "IF" HI > ABS(G[S[0]])   "THEN"
         "BEGIN" S[0]:= I;
            "IF" G[J]/G[S[N]] > 1 "THEN" S[N]:=J
         "END"
      "END" "ELSE"
      "IF" HI > ABS(G[S[N]]) "THEN"
      "BEGIN" S[N]:= "IF" G[J]/G[S[NM1]] > 1  "THEN" J "ELSE" S[NM1];
         "FOR" K:= NM1 "STEP" -1 "UNTIL" 1 "DO" S[K]:= S[K-1];
         S[0]:= I
      "END"
   "END" "ELSE"
   "BEGIN" "IF" HJ > MAX "THEN" "BEGIN" MAX:= HJ; INDEXMAX:= J "END";
      "IF" SIGN(G[J]) = SIGN(G[S[N]]) "THEN"
         "BEGIN" "IF" HJ > ABS(G[S[N]])  "THEN"
            "BEGIN" S[N]:= J; "IF" G[I]/G[S[0]] > 1 "THEN"S[0]:=I "END"
         "END" "ELSE"
         "IF" HJ > ABS(G[S[0]]) "THEN"
         "BEGIN" S[0]:= "IF" G[I]/G[S[1]] > 1 "THEN" I "ELSE" S[1];
            "FOR" K:= 1 "STEP" 1 "UNTIL" NM 1 "DO" S[K]:= S[K+1];
            S[N]:= J
         "END"
      "END" RANDGEBIEDEN;
      EM[1]:=MAX;
"END" SNOREMEZ;
       "EOP"
```

```
"CODE"36022;
 "PROCEDURE" MINMAXPOL(N,M,Y,FY,CO,EM);
 "VALUE" N,M;"INTEGER" N,M;
"ARRAY" Y,FY,CO,EM;
"COMMENT" MINMAXPOL CALCULATES THE COEFFICIENTS,
CO[I],I=,...N-1 OF THE POLYNOMIAL
P(Y)=CO[0]+CO[1]*Y+...+CO[N-1]*Y**(N-1),
THAT APPROXIMATES THE DISCRETE FUNCTION FY[I],I=0,...M,
GIVEN FOR THE ARGUMENTS Y[I],I=0,...M,
IN THE MINIMAX NORM.
THE ARGUMENTS MUST BE GIVEN IN MONOTONE INCREASING ORDER.
IN ARRAY EM[0:3], ONE MUST GIVE THE MAXIMUM ALLOWED NUMBER OF
ITERATIONS,EM[2].
MOREOVER,
EM[0]:=THE DIFFERENCE OF THE GIVEN FUNCTION AND THE POLYNOMIAL
IN THE FIRST APPROXIMATION POINT,
EM[1]:=THE MAXIMUM OF ! P(Y[I])-FY[I]!  FOR I=0,...M,
EM[3]:=THE NUMBER OF ITERATIONS PERFORMED.
THE PROCEDURES ELMVEC,ABSMAXVEC,POL,NEWTON,NEWGRN,
INI,SNDREMEZ
ARE USED.
REFERENCE:MEINARDUS,G.(1964,CH.7),
APPROXIMATION VON FUNKTIONEN UND IHRE NUMERISCHE BEHANDLUNG;
"BEGIN" "INTEGER" NM1,K,POMK,COUNT,CNT,J,MI;
"REAL" E,ABSE,ABSEH;
"INTEGER" "ARRAY" S[0:N];
"ARRAY" X,B[0:N]
,G[0:M];

"PROCEDURE" ELMVEC(L,U,SHIFT,A,B,X); "CODE" 34020;

"REAL" "PROCEDURE" ABSMAXVEC(K,LOW,UP,A); "CODE" 31060;


"REAL" "PROCEDURE" POL(N,X,A); "CODE" 31040;

"PROCEDURE" NEWTON(N,X,F); "CODE" 36010;

"PROCEDURE" NEWGRN(N,X,C); "CODE" 31050;

"PROCEDURE" INI(N,M,S); "CODE" 36020;

"PROCEDURE" SNDREMEZ(N,M,S,G,EM); "CODE" 36021;

"PROCEDURE" ERRPOL(N,M,E,CO,S,Y,FY,G);
"VALUE" N,M,E;"INTEGER" N,M;
"REAL" E;
"INTEGER" "ARRAY" S;"ARRAY" CO,Y,FY,G;                    "COMMENT"
```

```
"COMMENT"ERRPOL DELIVERS THE VALUE OF
CO[0]+CO[1]*Y[I]+...+CO[N-1]*Y[I]**(N-1) - FY[I]
IN G[I] FOR I=0,1,...M AND I NOT EQUAL S[J],J=0,1,...N.
FOR J=0,1,...N THEN G[S[J]]:=(-1)**J*E.
THE INTEGERS S[J],FOR J=0,1,...N ARE A SUBSET OF 0,1,...M;
 "BEGIN" "INTEGER" J,K,NM1,SJM1,SJ,S0,UP;
 NM1:=N-1;S0:=SJM1:=S[0];
 G[S0]:=E;
 "FOR" J:=1 "STEP" 1 "UNTIL" N "DO"
 "BEGIN" SJ:=S[J];UP:=SJ-1;
 "FOR" K:= SJM1+1 "STEP" 1 "UNTIL" UP "DO"
    G[K]:=FY[K]-POL(NM1,Y[K],CO);
 G[SJ]:=E:=-E;
 SJM1:=SJ;
 "END" J;
 "FOR" K:= S0-1 "STEP"-1 "UNTIL" 0 "DO"
 G[K]:=FY[K]- POL(NM1,Y[K],CO);
 "FOR" K:= SJ+1 "STEP" 1 "UNTIL" M "DO"
 G[K]:=FY[K]- POL(NM1,Y[K],CO);
 "END" ERRPOL;


INI(N,M,S);
NM1:=N-1;
MI:=EM[2];
ABSE:= 0;
"FOR" COUNT:= 1, COUNT + 1 "WHILE" COUNT <= MI & ABSE > ABSEH "DO"
"BEGIN"
POMK:=1;
"FOR" K:= 0 "STEP" 1 "UNTIL" N "DO"
"BEGIN" X[K]:= Y[S[K]]; CO[K]:= FY[S[K]]; B[K]:= POMK;
    POMK:=-POMK "END";
NEWTON(N,X,CO); NEWTON(N,X,B);
EM[0]:=
E:= CO[N]/B[N];
ELMVEC(0,NM1,0,CO,B,-E);
NEWGRN(NM1,X,CO);
ERRPOL(N,M,E,CO,S,Y,FY,G);
SNDREMEZ(N,M,S,G,EM);
   ABSEH:=ABSE;      ABSE:=ABS(E);
CNT:=COUNT;
"END" WHILE COUNT;
EM[2]:=MI;
EM[3]:=CNT;
"END" MINMAXPOL;
        "EOP"
```

SECTION : 7.2.1 (SEPTEMBER 1974)

AUTHOR: C.G. VAN DER LAAN.

INSTITUTE: MATHEMATICAL CENTRE.

RECEIVED: 730831.

BRIEF DESCRIPTION:

THIS SECTION CONTAINS TWO PROCEDURES:

INI SELECTS A (SUB)SET OF INTEGERS OUT OF A GIVEN SET OF INTEGERS.

SNDREMEZ EXCHANGES AT MOST N+1 NUMBERS WITH NUMBERS OUT OF A REFERENCE SET.

KEYWORDS:

ARGUMENTS EXTREME VALUES,
CHEBYSHEV POLYNOMIAL,
SECOND REMEZ ALGORITHM.

SUBSECTION : INI.

CALLING SEQUENCE:

THE HEADING OF THE PROCEDURE READS:

```
"PROCEDURE" INI(N,M,S);
"VALUE"N,M;"INTEGER"N,M""INTEGER""ARRAY"S;
```

THE MEANING OF THE FORMAL PARAMETERS IS:

N,M: <ARITHMETIC EXPRESSION>;
THE NUMBER OF POINTS TO BE SELECTED EQUALS N+1;
THE REFERENCE SET CONTAINS THE NUMBERS 0,1,...,M, M>=N;
S: <ARRAY IDENTIFIER>;
"INTEGER" "ARRAY" S[0:N];
EXIT:THE SELECTED INTEGERS ARE DELIVERED IN S.

PROCEDURES USED: NONE.

LANGUAGE: ALGOL 60.

METHODS AND PERFORMANCE:

THE ARGUMENTS FOR WHICH THE CHEBYSHEV POLYNOMIAL OF DEGREE N ATTAINS ITS EXTREME VALUES ON THE INTERVAL [-1,1] ARE TRANSFORMED TO THE INTERVAL [0,M] BY A LINEAR TRANSFORMATION;FINALLY THE NUMBERS ARE PROPERLY ROUNDED.

EXAMPLE OF USE:

```
"BEGIN""INTEGER""ARRAY"S[0:2];
"PROCEDURE"INI(N,M,S);"CODE"36020;
INI(2,20,S);
OUTPUT(61,"(""("INI SELECTS OUT OF 0,1,...,20 THE NUMBERS:")",/,
   3(B=ZDB)")",S[0],S[1],S[2])
"END"

INI SELECTS OUT OF 0,1,...,20 THE NUMBERS:
   0    10    20
```

SUBSECTION : SNDREMEZ.

CALLING SEQUENCE:

THE HEADING OF THE PROCEDURE READS:

```
"PROCEDURE"SNDREMEZ(N,M,S,G,EM);
"VALUE"N,M;"INTEGER"N,M;"INTEGER""ARRAY"S;"ARRAY" G,EM;
```

THE MEANING OF THE FORMAL PARAMETERS IS:

N,M: <ARITHMETIC EXPRESSION>;
THE NUMBER OF POINTS TO BE EXCHANGED IS SMALLER THAN OR EQUAL TO N+1;THE REFERENCE SET CONTAINS THE NUMBERS 0,1,...,M ,M>=N;

S: <ARRAY IDENTIFIER>;
"INTEGER" "ARRAY" S[0:N];
ENTRY:IN S ONE MUST GIVE N+1 (STRICTLY) MONOTONE INCREASING NUMBERS OUT OF 0,...,M;
EXIT :N+1 (STRICTLY) MONOTONE INCREASING NUMBERS OUT OF THE NUMBERS 0,1,...,M;

G: <ARRAY IDENTIFIER>;
"ARRAY" G[0:M];
ENTRY:IN ARRAY G[0:M] ONE MUST GIVE FUNCTIONVALUES;

EM: <ARRAY IDENTIFIER>;
"ARRAY" EM[0:1];
ENTRY:0<EM[0]<=G[I],I=0,...,M;
EXIT: EM[1]:=INFINITY NORM OF ARRAY G[0:M].

PROCEDURES USED:ABSMAXVEC=CP31060.

LANGUAGE: ALGOL 60.

METHOD AND PERFORMANCE:

THE SECOND REMEZ ALGORITHM IS USED (MEINARDUS,G.(1964)).

REFERENCE:

G.MEINARDUS,APPROXIMATION OF FUNCTIONS AND THEIR NUMERICAL TREATMENT(GERMAN),SPRINGER TRACTS IN NATURAL PHILOSOPHY,VOLUME 4, 1964.

EXAMPLE OF USE:

```
"BEGIN""ARRAY"EM[0:1],G[0:7];"INTEGER""ARRAY"S[0:2];
"PROCEDURE"SNDREMEZ(N,M,S,G,EM);"CODE"36021;
G[0]:=10;G[1]:=12;G[2]:=-15;G[3]:=-10;
G[4]:=-14;G[5]:=15;G[6]:=10;G[7]:=11;
EM[0]:=10;S[0]:=0;S[1]:=3;S[2]:=6;
OUTPUT(61,"("""("THE NUMBERS:")",/,"("S[J]:")",3(B-D),/,
    "("G[S[J]]:")",3(B-DD)")",
    S[0],S[1],S[2],G[S[0]],G[S[1]],G[S[2]]);
SNDREMEZ(2,7,S,G,EM);
OUTPUT(61,"("//,"("ARE EXCHANGED WITH:")",/,"("S[J]:")",3(B-D),/,
    "("G[S[J]]:")",3(B-DD),//,
    "("THE REFERENCE SET OF FUNCTIONVALUES IS:")",/,8(B-DD)")",
    S[0],S[1],S[2],G[S[0]],G[S[1]],G[S[2]] ,
    G[0],G[1],G[2],G[3],G[4],G[5],G[6],G[7])
"END"

THE NUMBERS:
S[J]:   0   3   6
G[S[J]]:   10 -10   10

ARE EXCHANGED WITH:
S[J]:   0   2   5
G[S[J]]:   10 -15   15

THE REFERENCE SET OF FUNCTIONVALUES IS:
  10   12 -15 -10 -14   15   10   11
```

SOURCE TEXT(S) :

```
"CODE"36020;
"PROCEDURE" INI(N,M,S);
"VALUE" N,M;"INTEGER" N,M;
"INTEGER" "ARRAY" S;
 "COMMENT" INI DELIVERS (MONOTONE) THE ROUNDED VALUES
 OF THE ARGUMENTS,WHERE THE CHEBYSHEV POLYNOMIAL
   OF DEGREE N(TRANSFORMED TO THE INTERVAL [0,M],M>=N)
 ATTAINS ITS MAXIMUM VALUES,
 IN INTEGER ARRAY S[0:N];
"BEGIN""INTEGER"I,J,K,L;"REAL"PIN2;
    PIN2:=ARCTAN(1)*2/N;
    K:=0;L:=N-1;J:=S[0]:=0;S[N]:=M;
 "FOR" K:=K+1 "WHILE" K < L "DO"
    "BEGIN"I:=SIN(K*PIN2)**2*M;
          J:=S[K]:="IF"I<=J"THEN"J+1"ELSE"I;
          S[L]:=M-J;L:=L-1
"END"K;
"IF"L*2=N"THEN"S[L]:=M/2;
 "END" INI;
        "EOP"

"CODE"36021;
 "PROCEDURE" SNDREMEZ(N,M,S,G,EM);
 "VALUE" N,M;"INTEGER" N,M;
 "INTEGER" "ARRAY" S; "ARRAY" G,EM;
 "COMMENT" SNDREMEZ EXCHANGES ATMOST N+1 NUMBERS ,GIVEN IN
           INTEGER ARRAY S[0:N], WITH NUMBERS OUT OF THE
           REFERENCE SET 0,...M, UNDER THE CONDITIONS:
      I.   THE ALTERNANCE PROPERTY OF THE FUNCTIONVALUES G[S[J]],
           J=0,...N IS PRESERVED,
      II.  |G[S[J]]|>=|EM[0]|,J=0,...N,
      III. THE FIRST INDEX K , WITH G[K]=INFINITY NORM OF G,
           IS ONE OF THE RESULTING NUMBERS S[0],...S[N],
           IN ARRAY G[0:M] ONE MUST GIVE ERROR FUNCTION VALUES,
           MOREOVER,
           EM[1]:=INFINITY NORM OF G,
     THE PROCEDURE ABSMAXVEC IS USED;
"BEGIN" "INTEGER" S0,SN,SJP1,I,J,K,UP,INDEXMAX,LOW,NM1;
   "REAL" MAX,MSJP1,HI,HJ,HE,ABSE,H;
   "REAL" "PROCEDURE" ABSMAXVEC(K,LOW,UP,A); "CODE" 31060;
                                                "COMMENT"
```

```
;

      INDEX MAX:=S0:=SJP1:=S[0];
      HE:=EM[0];LOW:=S0+1;
      MAX:=MSJP1:=ABSE:=ABS(HE);
      NM1:=N-1;
      "FOR" J:= 0 "STEP" 1 "UNTIL" NM1 "DO"
      "BEGIN"
         UP:= S[J+1]-1;
         H:= ABSMAXVEC(I,LOW,UP,G);
         "IF" H > MAX  "THEN" "BEGIN" MAX:= H; INDEX MAX:= I "END";
         "IF" H > ABSE "THEN"
         "BEGIN" "IF" HE * G[I] > 0 "THEN"
            "BEGIN" S[J]:= "IF" MSJP1 < H "THEN" I "ELSE" SJP1;
               SJP1:= S[J+1]; MSJP1:= ABSE
            "END" "ELSE"
            "BEGIN" S[J]:= SJP1; SJP1:= I; MSJP1:= H "END"
         "END" "ELSE"
         "BEGIN" S[J]:=SJP1; SJP1:=S[J+1]; MSJP1:= ABSE "END";
         HE:=-HE;LOW:=UP+2;
   "END" FOR J; SN:= S[N]; S[N]:= SJP1;

   HI:=ABSMAXVEC(I,0,S0-1,G);
   HJ:=ABSMAXVEC(J,SN+1,M,G);
   "IF" J  > M  "THEN" J:=M;
   "IF" HI > HJ "THEN"
   "BEGIN" "IF" HI > MAX "THEN" "BEGIN" MAX:= HI; INDEXMAX:= I "END";
      "IF" SIGN(G[I]) = SIGN(G[S[0]]) "THEN"
      "BEGIN" "IF" HI > ABS(G[S[0]])  "THEN"
         "BEGIN" S[0]:= I;
            "IF" G[J]/G[S[N]] > 1 "THEN" S[N]:=J
         "END"
      "END" "ELSE"
      "IF" HI > ABS(G[S[N]]) "THEN"
      "BEGIN" S[N]:= "IF" G[J]/G[S[NM1]] > 1  "THEN" J "ELSE" S[NM1];
         "FOR" K:= NM1 "STEP" -1 "UNTIL" 1 "DO" S[K]:= S[K-1];
         S[0]:= I
      "END"
   "END" "ELSE"
   "BEGIN" "IF" HJ > MAX "THEN" "BEGIN" MAX:= HJ; INDEXMAX:= J "END";
      "IF" SIGN(G[J]) = SIGN(G[S[N]]) "THEN"
         "BEGIN" "IF" HJ > ABS(G[S[N]])  "THEN"
            "BEGIN" S[N]:= J; "IF" G[I]/G[S[0]] > 1 "THEN"S[0]:=I "END"
         "END" "ELSE"
         "IF" HJ > ABS(G[S[0]]) "THEN"
         "BEGIN" S[0]:= "IF" G[I]/G[S[1]] > 1 "THEN" I "ELSE" S[1];
            "FOR" K:= 1 "STEP" 1 "UNTIL" NM 1 "DO" S[K]:= S[K+1];
            S[N]:= J
         "END"
      "END" RANDGEBIEDEN;
      EM[1]:=MAX;
"END" SNDREMEZ;
        "EOP"
```

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BRIEF DESCRIPTION:

MINMAXPOL CALCULATES THE COEFFICIENTS OF THE POLYNOMIAL (IN THE GRUNERT FORM) THAT APPROXIMATES A FUNCTION, GIVEN FOR DISCRETE ARGUMENTS, IN SUCH A WAY THAT THE INFINITY NORM OF THE ERROR VECTOR IS MINIMISED.

KEYWORDS:

MINIMAX POLYNOMIAL APPROXIMATION,
SECOND REMEZ EXCHANGE ALGORITHM.

CALLING SEQUENCE:

THE HEADING OF THE PROCEDURE READS:
"PROCEDURE"MINMAXPOL(N,M,Y,FY,CO,EM);
"VALUE"N,M;"INTEGER"N,M;"ARRAY"Y,FY,CO,EM;

THE MEANING OF THE FORMAL PARAMETERS IS:

N: <ARITHMETIC EXPRESSION>;
THE DEGREE OF THE APPROXIMATING POLYNOMIAL IS N=1;

M: <ARITHMETIC EXPRESSION>;
THE NUMBER OF REFERENCE FUNCTION VALUES VIZ, ARGUMENTS IS M+1;

Y,FY: <ARRAY IDENTIFIERS>;
"ARRAY"Y,FY[0:M];
ENTRY: FY[I] IS THE FUNCTION VALUE AT Y[I], FOR I=0,...,M;

CO: <ARRAY IDENTIFIER>;
"ARRAY"CO[0:N];
EXIT: THE COEFFICIENTS OF THE APPROXIMATING POLYNOMIAL, OF DEGREE N=1, ARE DELIVERED IN CO[0:N=1] (CO[N=1] IS COEFFICIENT OF Y**(N=1));

EM: <ARRAY IDENTIFIER>;
"ARRAY"EM[0:3];
ENTRY: EM[2]:THE MAXIMUM ALLOWED NUMBER OF ITERATIONS;
EXIT: EM[0]:THE DIFFERENCE OF THE GIVEN FUNCTION AND THE POLYNOMIAL IN THE FIRST APPROXIMATION POINT;
EM[1]:THE INFINITY NORM OF THE ERROR OF APPROXIMATION OVER THE DISCRETE INTERVAL;
EM[3]:THE NUMBER OF ITERATIONS PERFORMED.

PROCEDURES USED:

```
ELMVEC    =CP34020,
ABSMAXVEC=CP31060,
NEWTON    =CP36010,
POL       =CP31040,
NEWGRN    =CP31050,
INI       =CP36020,
SNDREMEZ  =CP36021.
```

REQUIRED CENTRAL MEMORY : ONE INTEGER ARRAY AND TWO REAL ARRAYS OF ORDER N + 1, AND ONE REAL ARRAY OF ORDER M + 1 ARE USED.

RUNNING TIME:

THE SECOND REMEZ ALGORITHM(ON A DISCRETE SET) IS QUADRATIC CONVERGENT;IN EACH ITERATION THE NUMBER OF OPERATIONS (MULTIPLICATIONS AND ADDITIONS) IS PROPORTIONAL TO M*N.

LANGUAGE: ALGOL 60.

METHOD AND PERFORMENCE: SEE REF.[1],CH.7.

REFERENCES:

[1].G.MEINARDUS.
APPROXIMATION OF FUNCTIONS AND THEIR NUMERICAL TREATMENT
(GERMAN).SPRINGER TRACTS IN NATURAL PHILOSOPHY.
VOLUME 4,1964.

EXAMPLE OF USE:

```
     "BEGIN""INTEGER"N;
     "PROCEDURE"MINMAXPOL(N,M,Y,FY,CO,EM); "CODE" 36022;
      "PROCEDURE" COMPUTE(N,A,B,F);
      "VALUE" N,A,B;"INTEGER" N;"REAL" A,B;
      "REAL" "PROCEDURE" F;
      "BEGIN" "INTEGER" K,L,M;
     "REAL"R,T,IDM;
      "ARRAY" COEF[0:N],EM[0:3];
      EM[2]:=10*N;
      M:=100*N;
      "BEGIN" "ARRAY" Y,FY[0:M];
      IDM:=(B-A)/M;
      R:=Y[0]:=A;FY[0]:=F(R);
      R:=Y[M]:=B;FY[M]:=F(R);
      L:=M-1;
      "FOR"K:=1"STEP"1"UNTIL"L"DO"
      "BEGIN"R:=Y[K]:=A+K*IDM;FY[K]:=F(R) "END";
      MINMAXPOL(N,M,Y,FY,COEF,EM);
      OUTPUT(61,"(""("COEF:")"/")");
      "FOR"K:=0"STEP"1"UNTIL"N-1"DO"OUTPUT(61,"(" ")",COEF[K]);
      OUTPUT(61,"("/8S/,2(N),2(B+3ZDB),/")","("EM[0:3]")",EM[0],EM[1],
      EM[2],EM[3]);
       "END";
       "END" COMPUTE;

      "REAL""PROCEDURE"F(X);"VALUE"X;"REAL"X;
      F:=1/(X-10);

       "FOR" N:=2"DO"
      "BEGIN" OUTPUT(61,"("//,"("DEGREE=")",D//")",N-1);
          COMPUTE(N,-1,1,F)
      "END"
       "END";


      DEGREE=1

      COEF:
      -1,0050378153393"-001   -1,0101010101010"-002
      EM[0:3]
      -5,0631947616870"-004   +5,0631947616870"-004      +20      +3
```

SOURCE TEXT(S):

```
"CODE"36022;
"PROCEDURE" MINMAXPOL(N,M,Y,FY,CO,EM);
"VALUE" N,M;"INTEGER" N,M;
"ARRAY" Y,FY,CO,EM;
"COMMENT" MINMAXPOL CALCULATES THE COEFFICIENTS,
CO[I],I=,...N-1 OF THE POLYNOMIAL
P(Y)=CO[0]+CO[1]*Y+...+CO[N-1]*Y**(N-1),
THAT APPROXIMATES THE DISCRETE FUNCTION FY[I],I=0,...M,
GIVEN FOR THE ARGUMENTS Y[I],I=0,...M,
IN THE MINIMAX NORM,
THE ARGUMENTS MUST BE GIVEN IN MONOTONE INCREASING ORDER,
IN ARRAY EM[0:3], ONE MUST GIVE THE MAXIMUM ALLOWED NUMBER OF
ITERATIONS,EM[2],
MOREOVER,
EM[0]:=THE DIFFERENCE OF THE GIVEN FUNCTION AND THE POLYNOMIAL
IN THE FIRST APPROXIMATION POINT,
EM[1]:=THE MAXIMUM OF | P(Y[I])-FY[I] |  FOR I=0,...M,
EM[3]:=THE NUMBER OF ITERATIONS PERFORMED,
THE PROCEDURES ELMVEC,ABSMAXVEC,POL,NEWTON,NEWGRN,
INI,SNDREMEZ
ARE USED,
REFERENCE:MEINARDUS,G.(1964,CH,7),
APPROXIMATION VON FUNKTIONEN UND IHRE NUMERISCHE BEHANDLUNG;
"BEGIN" "INTEGER" NM1,K,POMK,COUNT,CNT,J,MI;
"REAL" E,ABSE,ABSEH;
"INTEGER" "ARRAY" S[0:N];
"ARRAY" X,B[0:N]
,G[0:M];

"PROCEDURE" ELMVEC(L,U,SHIFT,A,B,X); "CODE" 34020;

"REAL" "PROCEDURE" ABSMAXVEC(K,LOW,UP,A); "CODE" 31060;


"REAL" "PROCEDURE" POL(N,X,A); "CODE" 31040;

"PROCEDURE" NEWTON(N,X,F); "CODE" 36010;

"PROCEDURE" NEWGRN(N,X,C); "CODE" 31050;

"PROCEDURE" INI(N,M,S); "CODE" 36020;

"PROCEDURE" SNDREMEZ(N,M,S,G,EM); "CODE" 36021;

"PROCEDURE" ERRPOL(N,M,E,CO,S,Y,FY,G);
"VALUE" N,M,E;"INTEGER" N,M;
"REAL" E;
"INTEGER" "ARRAY" S;"ARRAY" CO,Y,FY,G;                                    "COMMENT"
```

```
"COMMENT"ERRPOL DELIVERS THE VALUE OF
CO[0]+CO[1]*Y[I]+...+CO[N-1]*Y[I]**(N-1) - FY[I]
IN G[I] FOR I=0,1,...M AND I NOT EQUAL S[J],J=0,1,...N.
FOR J=0,1,...N THEN G[S[J]]:=(-1)**J*E.
THE INTEGERS S[J],FOR J=0,1,...N ARE A SUBSET OF 0,1,..,M;
"BEGIN" "INTEGER" J,K,NM1,SJM1,SJ,S0,UP;
NM1:=N-1;S0:=SJM1:=S[0];
G[S0]:=E;
"FOR" J:=1 "STEP" 1 "UNTIL" N "DO"
"BEGIN" SJ:=S[J];UP:=SJ-1;
"FOR" K:= SJM1+1 "STEP" 1 "UNTIL" UP "DO"
   G[K]:=FY[K]-POL(NM1,Y[K],CO);
G[SJ]:=E:=-E;
SJM1:=SJ;
"END" J;
"FOR" K:= S0-1 "STEP"-1 "UNTIL" 0 "DO"
G[K]:=FY[K]- POL(NM1,Y[K],CO);
"FOR" K:= SJ+1 "STEP" 1 "UNTIL" M "DO"
G[K]:=FY[K]- POL(NM1,Y[K],CO);
"END" ERRPOL;
```

```
INI(N,M,S);
NM1:=N-1;
MI:=EM[2];
ABSE:= 0;
"FOR" COUNT:= 1, COUNT + 1 "WHILE" COUNT <= MI & ABSE > ABSEH "DO"
"BEGIN"
POMK:=1;
"FOR" K:= 0 "STEP" 1 "UNTIL" N "DO"
"BEGIN" X[K]:= Y[S[K]]; CO[K]:= FY[S[K]]; B[K]:= POMK;
    POMK:=-POMK "END";
NEWTON(N,X,CO); NEWTON(N,X,B);
EM[0]:=
E:= CO[N]/B[N];
ELMVEC(0,NM1,0,CO,B,-E);
NEWGRN(NM1,X,CO);
ERRPOL(N,M,E,CO,S,Y,FY,G);
SNDREMEZ(N,M,S,G,EM);
   ABSEH:=ABSE;      ABSE:=ABS(E);
CNT:=COUNT;
"END" WHILE COUNT;
EM[2]:=MI;
EM[3]:=CNT;
"END" MINMAXPOL;
        "EOP"
```